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On the distribution of zeros of a sequence of entire functions approaching the Riemann zeta function

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ABSTRACT

In this paper we study the distribution of zeros of each entire function of the sequence $\{G_n(z) \equiv 1 + 2^z + \cdots + n^z\colon\ n \geqslant 2\}$, which approaches the Riemann zeta function for $\operatorname{Re} z < -1$, and is closely related to the solutions of the functional equations $f(z) + f(2z) + \cdots + f(nz) = 0$. We determine the density of the zeros of $G_n(z)$ on the critical strip where they are situated by using almost-periodic functions techniques. Furthermore, by using a theorem of Kronecker, we also establish a formula for the number of zeros of $G_n(z)$ inside certain rectangles in the critical strip.

1. Introduction

In 1782 Lagrange connected the astronomical problem of the perturbation of the large planets with the variation of the argument of certain exponential polynomials, emerging a mathematical problem known as Mean Motion [4]. To extend Lagrange's problem from exponential polynomials to a more general class, H. Bohr [2] in 1924 introduced the notion of almost-periodic function, a powerful theory that we will use through the present paper, following [1] and [5], to study the asymptotic distribution of zeros of $G_n(z)$.

In [7], for each integer $n \geqslant 2$, we connected the functional equation

$$f(x) + f(2x) + \cdots + f(nx) = 0, \quad x > 0, \tag{1.1}$$

with the exponential polynomial

$$1 + 2^z + \cdots + n^z \tag{1.2}$$

in such a way that the continuous solutions of (1.1) depend on the zeros of (1.2). About the above functional equation we can say that it is as interesting from an applied point of view as from a theoretical one. Indeed, Eq. (1.1), for the case $n = 2$, yields continuous solutions on $\mathbb{R} \setminus \{0\}$ simulating the planar motion of a particle between two parallel sheets in such a way that the velocity at a point, say $2x$, is reduced to half with change of sign, with respect to the velocity at the point x, for any $x \neq 0$. In the case $n = 3$, (1.1) has continuous solutions on whole $\mathbb{R}$ whose graphs are limited by the equations

$$y = \pm x^a, \quad a > 0,$$

with $a = \operatorname{Re} \beta_3$, where β_3 is a zero of the exponential polynomial

$$1 + 2^z + 3^z.$$

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Moreover, in [6], we can find a proof, for $n = 2, 3$, on the existence of certain compact subsets of the plane which are "almost filled" by the continuous solutions of (1.1). This property is another interesting peculiarity of this special functional equation, and it differs significantly from the Cauchy functional equation. Anyway, for arbitrary n, the existence of solutions of (1.1) has been treated in [7, Theorem 6], where a proof by using Euler's operator was given to establish the connection between (1.1) and (1.2). In this way, the above considerations allow us to justify the interest to study the distribution of the zeros of the exponential polynomial (1.2), for arbitrary $n \geqslant 2$, from now on designated by

$$G_n(z) \equiv 1 + 2^z + \cdots + n^z.$$

For each integer $n \geqslant 2$ one can easily check that $G_n(z)$ is an entire function of order 1, exponential type $\sigma = \ln n$. It will be shown that $G_n(z)$ has its zeros in a strip parallel to the imaginary axis, called critical strip. Another peculiarity of the functions $G_n(z)$ is that, except when $n = 2$, they have at least a zero which does not lie on the imaginary axis [7]. However, as we will show below, the functions $G_n(z)$ have "almost all" the zeros in a "neighbourhood" of the imaginary axis.

In the present paper we are specially interested, firstly, to study the asymptotic distribution of the zeros of the functions $G_n(z)$ in each critical strip. Secondly, for each n, our interest is focused on the existence of infinitely many rectangles, say $R_{n,T}$, formed by the critical strip bounded by the lines $\operatorname{Im} z = 0$, $\operatorname{Im} z = T$, for which we can find an exact formula to determine the number of zeros of $G_n(z)$ inside each rectangle.

2. The zeros of $G_n(z)$ from the almost-periodic theory

The case $n = 2$ is specially easy, all the zeros of $G_2(z)$ are simple and all of them lie on the imaginary axis. For $n > 2$, the following result settles an upper bound for the order of multiplicity of the zeros of $G_n(z)$.

Proposition 1. *For $n > 2$, any zero of $G_n(z) \equiv 1 + 2^z + \cdots + n^z$ has multiplicity at most $n - 2$.*

Proof. Assume z_0 is a zero of $G_n(z)$ with multiplicity $n - 1$, then

$$G_n(z_0) = G_n'(z_0) = G_n''(z_0) = \cdots = G_n^{(n-2)}(z_0) = 0,$$

which is equivalent to the existence of solution of the system

$$\left.\begin{array}{l} 2^{z_0} + \cdots + n^{z_0} = -1, \\ 2^{z_0} \ln 2 + \cdots + n^{z_0} \ln n = 0, \\ \vdots \\ 2^{z_0} (\ln 2)^{n-2} + \cdots + n^{z_0} (\ln n)^{n-2} = 0. \end{array}\right\} \tag{2.1}$$

Taking $2^{z_0}, \ldots, n^{z_0}$ as unknowns, the determinant of its corresponding matrix in (2.1) is a non-null real Vandermonde. Then (2.1) has a unique real solution for $2^{z_0}, \ldots, n^{z_0}$ and, in particular, the numbers 2^{z_0}, 3^{z_0} would be real, which implies that z_0 is real. However, since $G_n(z) > 0$ for any real z, it contradicts our assumption. Hence, the proposition follows. □

The function $G_n(z)$ can be written as

$$\sum_{k=1}^{n} a_k e^{\lambda_k z},$$

with $a_k = 1$ and $\lambda_k = \ln k$, for $1 \leqslant k \leqslant n$. Then, according to the λ_k are points of the boundary of a bounded convex complex region, namely, the closed real segment $[0, \ln n]$, $G_n(z)$ is a function of completely regular growth. Hence, there exists an asymptotic formula, for the distribution of zeros of $G_n(z)$, which will be deduced from the theory of almost-periodic functions. To do it, we define the function

$$H_n(z) \equiv G_n(iz).$$

Then, with the usual notation Z_f for the set of zeros of a function f, one has

$$Z_{G_n} = i Z_{H_n},$$

and, consequently, the zeros of $G_n(z)$ will be obtained multiplying by i the zeros of $H_n(z)$.

On the other hand, for real x, the function

$$H_n(x) = \sum_{k=1}^{n} a_k e^{i\lambda_k x},$$

$a_k = 1$, $\lambda_k = \ln k$, for $1 \leqslant k \leqslant n$, is a generalized trigonometric polynomial and, consequently, an almost-periodic function on $\mathbb{R}$. Thus [1,5], there exist a mean defined by

$$M\{H_n\} \equiv \lim_{L\to\infty} \frac{1}{2L} \int_{-L+\alpha}^{L+\alpha} H_n(t)\,dt,$$

where the convergence is uniform relative to α, and the Fourier coefficients

$$a(\lambda) \equiv M\left\{H_n(x)e^{-i\lambda x}\right\}$$

are given by

$$a(\lambda) = \begin{cases} 1, & \text{for } \lambda = \ln k,\ 1 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the spectrum, S_{H_n}, of $H_n(x)$ is the finite set

$$S_{H_n} = \{\ln k\colon\ 1 \leqslant k \leqslant n\}.$$

Now, since the upper and lower bound of S_{H_n} enter into the spectrum, by applying a well-known theorem of H. Bohr [5, p. 270], all the roots of $H_n(z)$ are situated in some strip parallel to the real axis. Consequently, all the zeros of $G_n(z)$ are in a strip parallel to the imaginary axis. From now on, the strips where the zeros of $G_n(z)$ or $H_n(z)$ are situated, respectively, will be merely called critical strips.

We point out that the property of $G_n(z)$, $H_n(z)$, that all their zeros are in a critical strip, can be also proved, as in [7], by using the theorem of Rouché.

Recall that an entire function f with zeros $\{a_k\}$ satisfying

$$\sum_{k=1}^{\infty} \left| \operatorname{Im}\left(\frac{1}{a_k}\right) \right| < \infty$$

is called a function of class $\mathcal{A}$ [5, p. 222]. That is, the class of those entire functions that have "almost all" the zeros in a "neighbourhood" of the real axis. In the next result we will prove that $H_n(z)$ is of class $\mathcal{A}$ and, likewise, we will deduce the existence of density.

Theorem 2. *Let $N_{H_n}(r)$ denote the number of zeros of $H_n(z)$ in $|z| < r$. Then the density associated with the function $H_n(z)$,*

$$\Delta_{H_n} \equiv \lim_{r\to\infty} \frac{N_{H_n}(r)}{r},$$

exists.

Proof. Since $H_n(z)$ is an entire function of order 1 and exponential type $\sigma = \ln n$, the number α defined by the formula

$$\alpha \equiv \lim\inf_{r\to\infty} \left(r^{-1} \log M_{H_n}(r)\right)$$

satisfies the inequality

$$\alpha \leqslant \ln n$$

and so α is finite. On the other hand, as

$$\left|H_n(x)\right| \leqslant n,$$

it is clear that the integral

$$\int_{-\infty}^{\infty} \frac{\log^+ |H_n(x)|}{1+x^2}\,dx < \infty, \tag{2.2}$$

where, as usual, for t real

$$\log^+ t \equiv \begin{cases} \ln t, & \text{if } t \geqslant 1, \\ 0, & \text{if } t < 1. \end{cases}$$

Hence, taking into account that α is finite, inequality (2.2) allows us to apply Pfluger's theorem [1, Theorem 6.3.6] to get

$$\int_{-\infty}^{\infty} \frac{|\log|H_n(x)||}{1+x^2}\,dx < \infty. \tag{2.3}$$

Now, according to

$$H_n(-x) = \overline{H_n(x)},$$

it follows

$$\left|H_n(x)\right|^2 = H_n(x)H_n(-x)$$

and, because of (2.3), we conclude

$$\int_0^{\infty} \frac{\log|H_n(x)H_n(-x)|}{1+x^2}\,dx < \infty. \tag{2.4}$$

On the other hand, since the entire function $H_n(z)$ is of exponential type and bounded on the real axis and it is clear that, for arbitrary $R > 1$,

$$\int_1^{R} \frac{\log|H_n(x)H_n(-x)|}{x^2}\,dx$$

is bounded above; noticing [1, p. 134], it follows that $H_n(z)$ is of class $\mathcal{A}$. Now, by applying [5, p. 246], the inequality (2.4) implies the existence of density. This completes the proof. □

The preceding theorem shows the existence of density. But it remains to determine its value, which will allow us to find an asymptotic estimation of the number of zeros in the critical strip with imaginary part between 0 and any positive real number T.

Theorem 3. *The number of zeros, $N_{G_n}(0,T)$, of the function $G_n(z)$ in the critical strip bounded by the lines $y=0$ and $y=T$, satisfies*

$$\lim_{T\to\infty} \frac{N_{G_n}(0,T)}{T} = \frac{\ln n}{2\pi}.$$

Proof. Consider two real numbers $y_1 < 0 < y_2$ such that the critical strip of $H_n(z)$, say S_{C,H_n}, to be contained in the strip

$$S(y_1, y_2) \equiv \{z\colon\ y_1 < \operatorname{Im} z < y_2\}. \tag{2.5}$$

Let us denote by $N_{H_n}(0,T,y_1,y_2)$ the number of zeros of $H_n(z)$ in the rectangle $[0,T]\times[y_1,y_2]$. According to

$$H_n(z) = \overline{H_n(-\bar{z})}$$

and taking into account the previous theorem, we can assure the existence of the limit

$$\lim_{T\to\infty} \frac{N_{H_n}(0,T,y_1,y_2)}{T} \equiv \sigma(y_1,y_2). \tag{2.6}$$

On the other hand, it is obvious that $\sigma(y_1,y_2)$ does not depend on y_1, y_2 provided that S_{C,H_n} to be contained in the strip defined in (2.5). Thus the limit in (2.6), as $y_1 \to -\infty$, $y_2 \to \infty$, exists and satisfies

$$\lim_{y_1\to-\infty;\ y_2\to+\infty} \sigma(y_1,y_2) = \lim_{T\to\infty} \frac{N_{H_n}(0,T)}{T}, \tag{2.7}$$

where $N_{H_n}(0,T)$ denotes the number of zeros of $H_n(z)$ in the critical strip satisfying $0 \leqslant \operatorname{Re} z \leqslant T$. Now, from [5, p. 286] and noticing $\ln n$ is the length of the smallest segment containing to the spectrum of $H_n(x)$, the value of the limit in (2.7) is exactly

$$\frac{\ln n}{2\pi}. \tag{2.8}$$

Finally, taking into account the relation between the functions $H_n(z)$ and $G_n(z)$, the theorem follows. □

3. The zeros of $G_n(z)$ from the argument principle

Observe that (2.8) is an asymptotic estimation, of the number of zeros of $G_n(z)$, obtained by using the sophisticated theory of H. Bohr [2] on almost-periodic functions. However,

$$G_n(z) = 1 + 2^z + \cdots + n^z$$

is a very nice function that deserves to be studied in a direct manner to find the distribution of its zeros, as we will do below. In this manner, we will prove that there exist infinitely many values of T, say privileged T's, for which we will establish a formula for the number of zeros of $G_n(z)$, $N_{G_n}(0, T)$, inside the rectangles in the critical strip bounded by the lines $y = 0$ and $y = T$. This result will considerably improve the asymptotic estimation achieved in the previous theorem.

We start with the determination of the exact number of zeros of $G_2(z)$ in the strip $0 \leqslant \operatorname{Im} z = y \leqslant T$, denoted by $N_2(T)$. In this case as the solutions of the equation

$$1 + 2^z = 0$$

are

$$z = \frac{(2k-1)\pi i}{\ln 2},$$

with k in the set $\mathbb{Z}$ of the integers, a direct computation gives us the formula

$$N_2(T) = \max\left\{k \in \mathbb{Z}\colon\ \frac{(2k-1)\pi}{\ln 2} \leqslant T\right\}$$

or, equivalently,

$$N_2(T) = \left[\frac{T \ln 2}{2\pi} + \frac{1}{2}\right], \tag{3.1}$$

where [ ] designates the integer part.

We now look for a formula similar to that of (3.1) for arbitrary $n \geqslant 3$. Firstly, observe that, for real z, the function $G_n(z)$ tends to 1 and ∞ as z goes to $-\infty$ and ∞, respectively. Thus we can determine, for each fixed n, real numbers r_n and s_n with $r_n < s_n$ such that

$$\left|G_n(z) - 1\right| < 1, \quad \text{for all } z \text{ with } \operatorname{Re} z \leqslant r_n \tag{3.2}$$

and

$$\left|\frac{G_n(z)}{n^z} - 1\right| < 1, \quad \text{for all } z \text{ with } \operatorname{Re} z \geqslant s_n, \tag{3.3}$$

concluding that all the zeros of $G_n(z)$ are in the vertical strip bounded by the lines

$$\operatorname{Re} z = r_n, \qquad \operatorname{Re} z = s_n.$$

Let T be a positive number, which will be specified with a much more precision below, and, for each function $G_n(z)$, the rectangle $R_{n,T}$ defined by the lines $x = r_n$, $x = s_n$ and $y = 0$, $y = T$. Without loss of generality, we may assume $G_n(z)$ has no zero on the horizontal line $y = T$. Then, since $G_n(z)$ does not have real zeros, from (3.2) and (3.3), we can assure that there are no zeros on the rectangle $R_{n,T}$. Therefore we can study the variation of the argument of $G_n(z)$ on $R_{n,T}$, from now on denoted by $\Theta_{n,T}$, starting from and returning to the point $(r_n, 0)$.

Observe that the main problem about the study of $\Theta_{n,T}$ on the rectangle $R_{n,T}$ is on the side defined from the point (s_n, T) to (r_n, T), which will be denoted by $R_{n,T,3}$. To avoid the complicated situation created by the possibility of vanishing either the real or imaginary part of $G_n(z)$ on $R_{n,T,3}$, we will use a consequence of an old result of Kronecker [3] which, in turn, requires the following trivial lemma.

Lemma 4. *Let n be an integer greater than 2 and $\{p_1, p_2, \ldots, p_{k_n}\}$ the set of all prime numbers less than or equal to n. Then the set $\{\ln p_1, \ln p_2, \ldots, \ln p_{k_n}\}$ is linearly independent, that is, any linear combination $n_1 \ln p_1 + n_2 \ln p_2 + \cdots + n_{k_n} \ln p_{k_n} = 0$, with integers $n_1, n_2, \ldots, n_{k_n}$, implies $n_1 = n_2 = \cdots = n_{k_n} = 0$. Furthermore, there exist integers $\{l_{mj}\colon\ m = 2, \ldots, n;\ j = 1, 2, \ldots, k_n\}$ such that*

$$\ln m = \sum_{j=1}^{k_n} l_{mj} \ln p_j \tag{3.4}$$

for each $m = 2, \ldots, n$.

Theorem 5 *(Kronecker–Bohl). Let $a_1, a_2, \ldots, a_k$ be non-null real numbers such that $a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1}$ are linearly independent. For arbitrary real numbers $b_1, b_2, \ldots, b_k$ and $\epsilon > 0$, there exists $l > 0$ (l only depends on $a_1, a_2, \ldots, a_k$ and ϵ) such that any interval of length l contains an interval of length ϵ that meets each set*

$$A_j = \{a_j p + b_j \colon\ p \in \mathbb{Z}\},$$

for $j = 1, \ldots, k$.

Our general formula to determine the number of zeros in each rectangle $R_{n,T}$ is given in the following result.

Theorem 6. *There exists a positive real number l such that each interval $(pl, (p+1)l)$, $p \in \mathbb{Z}$, $p \geqslant 0$, contains an interval I_p such that for any $T \geqslant \frac{2\pi}{\ln n}$, $T \in I_p$, the number of zeros of $G_n(z)$ inside the rectangle $R_{n,T}$ is given by the formula*

$$N_n(T) = \left[\frac{T \ln n}{2\pi} + \Omega_n\right], \tag{3.5}$$

with $|\Omega_n| < 1$.

Proof. (Based on the variation of the argument on $R_{n,T}$.) Noticing (3.1), observe that the formula (3.5) is verified for $n = 2$. Therefore, we can suppose $n > 2$.

Because $G_n(z)$ is real for real z, on the side determined by the points $(r_n, 0)$ and $(s_n, 0)$, denoted by $R_{n,T,1}$, the variation of the argument

$$\Theta_{n,T,1} = 0. \tag{3.6}$$

On the side determined by the points $(s_n, 0)$ and (s_n, T), denoted by $R_{n,T,2}$, the variation $\Theta_{n,T,2}$ can be estimated by writing

$$G_n(z) = n^z \frac{G_n(z)}{n^z} = n^{s_n} e^{iy \ln n} \frac{G_n(z)}{n^z}, \quad 0 \leqslant \operatorname{Im} z = y \leqslant T.$$

Now, taking into account (3.3), there exists some $\Phi_{n,2}$, which does not depend on T, such that

$$\Theta_{n,T,2} = T \ln n + \Phi_{n,2}, \tag{3.7}$$

with $|\Phi_{n,2}| < \pi$.

On the side determined by the points (s_n, T) and (r_n, T), denoted by $R_{n,T,3}$, we claim that the variation of the argument $\Theta_{n,T,3}$ satisfies

$$|\Theta_{n,T,3}| < \eta, \tag{3.8}$$

for arbitrarily small $\eta > 0$. Indeed, let $F_n(z)$ be the real part of the function $G_n(z) - 1$, that is,

$$F_n(z) = F_n(x + iy) = e^{x \ln 2} \cos(y \ln 2) + \cdots + e^{x \ln n} \cos(y \ln n).$$

Noticing (3.4), $F_n(z)$ becomes a polynomial in

$$e^{x \ln p_j}, \ \cos(y \ln p_j), \ \sin(y \ln p_j),$$

where $\{p_1, p_2, \ldots, p_{k_n}\}$ is the set of all prime numbers less than or equal to n, $j = 1, \ldots, k_n$. Thus, by continuity, given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in [r_n, s_n]$ one has

$$\left|F_n(x + iy) - F_n(x)\right| < \epsilon, \tag{3.9}$$

provided that

$$\left|\cos(y \ln p_j) - 1\right| < \delta, \qquad \left|\sin(y \ln p_j)\right| < \delta$$

for all $j = 1, \ldots, k_n$.

On the other hand, since for real z, $F_n(z) > 0$, there exists a positive constant A such that

$$F_n(x) \geqslant A > 0, \tag{3.10}$$

for any x in the real segment $[r_n, s_n]$. Pick an arbitrary positive constant $B < A$ and determine ϵ, less than $A - B$, and so small that for any numbers $\{\epsilon_j \colon\ 1 \leqslant j \leqslant k_n\}$ with $|\epsilon_j| \leqslant \epsilon$,

$$\left|\cos(\epsilon_j \ln p_j) - 1\right| < \delta, \qquad \left|\sin(\epsilon_j \ln p_j)\right| < \delta \tag{3.11}$$

holds.

Let us take $a_j = \frac{2\pi}{\ln p_j}$, $b_j = 0$, $1 \leqslant j \leqslant k_n$, and ϵ verifying (3.9). Now, by applying Kronecker's theorem, there exists a positive real number l such that any interval of the form $(pl, (p+1)l)$, $p \in \mathbb{Z}$, contains an interval I_p of length ϵ which contains at least one point of each set A_j, $1 \leqslant j \leqslant k_n$, defined by

$$A_j \equiv \left\{ \frac{2\pi q}{\ln p_j} \colon q \in \mathbb{Z} \right\}.$$

Hence, given any positive number T belonging to any interval I_p, $p \geqslant 0$, there exists some $c_j \in I_p \cap A_j$; setting $\epsilon_j = T - c_j$ and according to (3.11) one has

$$\left|\cos(T \ln p_j) - 1\right| < \delta, \qquad \left|\sin(\epsilon_j \ln p_j)\right| < \delta \tag{3.12}$$

for all $1 \leqslant j \leqslant k_n$. Taking into account (3.9) and (3.10), from (3.12) it follows that

$$F_n(x + iT) = F_n(x) + \big(F_n(x + iT) - F_n(x)\big) \geqslant A - \epsilon = B > 0,$$

concluding that

$$\operatorname{Re} G_n(z) > 1 \tag{3.13}$$

for any $z = x + iT$ of the side $R_{n,T,3}$.

On the other hand, as

$$\operatorname{Im} G_n(z) = e^{x \ln 2} \sin(y \ln 2) + \cdots + e^{x \ln n} \sin(y \ln n),$$

(3.12) also implies that $|\operatorname{Im} G_n(z)|$ is arbitrarily small on $R_{n,T,3}$ and, from (3.13), the variation of the argument on the side $R_{n,T,3}$ verifies (3.8), provided that T is chosen so that it belongs to any interval I_p, $p \geqslant 0$.

On the side determined by the points (r_n, T) and $(r_n, 0)$, denoted by $R_{n,T,4}$, it directly follows, from (3.2), that the variation of the argument $\Theta_{n,T,4}$ does not depend on T and satisfies

$$|\Theta_{n,T,4}| < \pi. \tag{3.14}$$

Finally, from (3.6), (3.7), (3.8) and (3.14), we conclude that the total variation of the argument on the rectangle $R_{n,T}$ satisfies

$$\Theta_{n,T} = T \ln n + \Psi_n, \quad |\Psi_n| < 2\pi,$$

and, consequently, the number of zeros of $G_n(z)$ situated in the interior of $R_{n,T}$ verifies the desired formula. $\square$

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